

Generalised Algebraic Axiomatisations of Contextual Categories

Corrected 6 April 2018

Extended 14 June 2018

Corrected 5 July 2018

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1 Introduction

The syntactic notion of *generalised algebraic theory* (GAT) and the equivalent algebraic notion of *contextual category* were introduced in [1] and summarised in [2]. Whereas generally the objects of categories can be taken to represent types, the objects of contextual categories can be taken to represent what I have described as ‘types that vary’ but which subsequently have become known as dependent types. Equally they can be taken to represent contexts: the algebra of contextual categories is an algebra of context.

In [1] it was noted that the theory of contextual categories can be expressed as a generalised algebraic theory by the axiomatisation of identity types on the morphism types of the category (the identity types are required in order to phrase the pullback structure required by the definition of contextual category); thus contextual categories are a generalised algebraic equivalent of GATs and, in this sense, the machinery of GATs is able to self-describe; something which cannot be said of Lawvere algebraic theories nor of many-sorted algebraic theories.

Voevodsky in [3] has shown a better way of representing contextual categories as generalised algebraic; one which doesn’t necessitate the introduction of types other than those for objects and morphisms. To do this he introduces an operator ‘ s ’ which canonically maps any morphism to a representative section; he shows that suitably axiomatised this implies the existence of pullbacks. The purpose of this paper is to describe the generalised algebraic theory of contextual categories which results from following this approach (note however that Voevodsky used the term *c-system* in preference to *contextual category*) and then to introduce a further generalised algebraic axiomatisation using operators $*$ and δ in place of Voevodsky’s operator s .

As a prelude to presenting the theory we first present two simpler but related theories (previously presented in [1] and [2]): the GAT of categories and the GAT of trees.

2 Background Theories and Notations

2.1 The GAT of Categories

Symbol *Introductory Rule*

Ob Ob is a type

Hom $x, y \in Ob \vdash Hom(x, y)$ is a type

\circ $x, y, z \in Ob, f \in Hom(x, y), g \in Hom(y, z) \vdash \circ(f, g) \in Hom(x, z)$

id $x \in Ob \vdash id(x) \in Hom(x, x)$

Axioms

$\circ(id(x), f) = f$, whenever $x, y \in Ob, f \in Hom(x, y)$

$\circ(f, id(y)) = f$, whenever $x, y \in Ob, f \in Hom(x, y)$

$\circ(\circ(f, g), h) = \circ(f, \circ(g, h))$, whenever $w, x, y, z \in Ob, f \in Hom(w, x), g \in Hom(x, y), h \in Hom(y, z)$

2.2 Trees of Concepts and the GAT of Trees

A contextual category has a tree of objects and we think of this tree as a tree of dependencies among concepts.

Formally, a tree is any partially ordered set $(S, <)$ such that for each $t \in S$, the set $\{s \in T : s < t\}$ is well-ordered by the relation $<$. We wish to consider just trees $(S, <)$ such that for each $t \in S$, the set $\{s \in S : s < t\}$ is finite and such that there is a unique root to the tree i.e. a unique least element. We call the elements of set S the nodes of the tree and for each node the cardinality of the set $\{s \in S : s < t\}$ is said to be the height of the node t . We denote by $S_i, i \geq 0$, the set of elements of S of height i . The set S_0 is a singleton set containing the root of the tree.

Such trees as these we can equivalently describe as models of the generalised algebraic theory given below table 1 in which the nodes of height $n + 1$ are represented as of a sort S_{n+1} that is dependent on the sort of nodes of height n .

If A and B are nodes of a tree $(S, <)$ then we shall write $A \triangleleft B$ to mean that $A < B$ in S and that there does not exist x such that $A < x < B$. For every node B of tree S other than the root node there exists a unique node A such that $A \triangleleft B$.

Table 1: The Generalised Algebraic Theory of Trees

<i>Symbol</i>	<i>Introductory Rule</i>
S_0	S_0 is a type
S_1	$x_0 \in S_0 \vdash S_1(x_0)$ is a type
S_2	$x_0 \in S_0, x_1 \in S_1(x_0) \vdash S_2(x_0, x_1)$ is a type
\vdots	
S_n	$x_0 \in S_0, x_1 \in S_1(x_0), \dots, x_{n-1} \in S_{n-1}(x_0, x_1, \dots, x_{n-2}) \vdash S_n(x_0, x_1, \dots, x_{n-1})$ is a type
\vdots	
<i>root</i>	<i>root</i> $\in S_0$
<i>Axioms:</i>	
$x, y \in S_0 \vdash x = y$	

2.3 Schematic Notation

There is a shorthand that is convenient in the presentation of the GAT of trees and then subsequently in the GAT of contextual categories. We use the shorthand $x \in S_n$ for the context $x_0 \in S_0, x_1 \in S_1(x_0), \dots, x_n \in S_n(x_0, x_1, \dots, x_{n-1})$.

Using this shorthand, for any $n \geq 0$ the sort S_n in the theory of trees is introduced as follows:

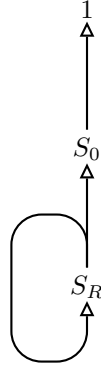
<i>Symbol</i>	<i>Introductory Rule</i>
S_0	S_0 is a type
$S_{n+1}, n \geq 0$	$x \in S_n \vdash S_{n+1}(x)$ is a type

2.4 An Aside on Recursive Type Definitions

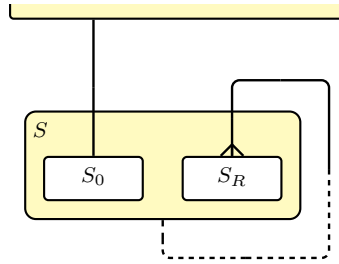
Using the shorthand, we are quite close to having a recursive definition of a single sort S_R . Such definitions are not possible in generalised algebraic theories but we can imagine a framework in which it is possible to write:

<i>Symbol</i>	<i>Introductory Rule</i>
S_0	S_0 is a type
S_R	$x_0 \in S_0 \vdash S_R(x_0)$ is a type
S_R	$x \in S_R \vdash S_R(x)$ is a type

Such a definition could be represented algebraically in a suitably generalised notion of contextual category (a comulti-contextual category?) these dependencies could be represented as follows:



This is not just an idle thought – in data modelling such a tree structure is represented in an entity model diagram in which the injections into the coproduct S of S_0 and S_R are represented by containment:

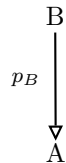


See www.entitymodelling.org/tutorialone for a description of this notation. For an example of the modelling of recursive relationships in the definition of a phrase structure grammar of English see www.entitymodelling.org/examplesone/englishsentence.

3 Definition of Contextual Categories

3.1 The GAT of Tree-Structured Categories

By a tree-structured category we mean (i) a category with a tree-structure defined on its objects such that the tree of objects has a unique root object and (ii) for every $A \triangleleft B$ in the tree of objects a canonical morphism $p_B : B \rightarrow A$. This morphism will be distinguished in diagrams by an arrow with a triangular head so:



The theory of tree-structured categories can be presented as a generalised algebraic theory as follows:

<i>Symbol</i>	<i>Introductory Rule</i>
Ob_0	Ob_0 is a type
Ob_{n+1}	$x \in Ob \vdash Ob_{n+1}(x)$ is a type
$Hom_{n,m}$	$x \in Ob, y \in Ob \vdash Hom_{n,m}(x, y)$ is a type
\circ	$x \in Ob, y \in Ob, z \in Ob, f \in Hom_{n,m}(x, y), g \in Hom_{m,p}(y, z) \vdash \circ(f, g) \in Hom_{n,p}(x, z)$
id_n	$x \in Ob \vdash id_n(x) \in Hom_{n,n}(x, x)$
p_n	$x \in Ob \vdash p_n(x) \in Hom_{n,n-1}(x, x_{n-1})$
1	$1 \in Ob_0$
t_n	$x \in Ob \vdash t_n(x) \in Hom_{n,0}(x, 1)$

Axioms

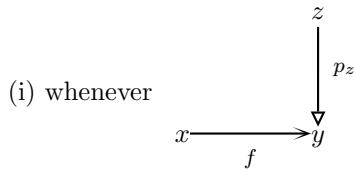
- $\circ(id(x), f) = f$, whenever $x \in Ob, y \in Ob, f \in Hom_{n,m}(x, y)$
 - $\circ(f, id(y)) = f$, whenever $x \in Ob, y \in Ob, f \in Hom_{n,m}(x, y)$
 - $\circ(\circ(f, g), h) = \circ(f, \circ(g, h))$, whenever
- $$w \in Ob, x \in Ob, y \in Ob, z \in Ob, f \in Hom_{l,n}(w, x), g \in Hom_{n,m}(x, y), h \in Hom_{m,p}(y, z)$$
- $x = y$, whenever $x, y \in Ob_0$
 - $f = t_n(x)$, whenever $x \in Ob, f \in Hom_{n,0}(x, 1)$

For ease of reading in what follows we omit the numeric subscripts and also use an infix notation, writing $f \circ g$ in place of $\circ(f, g)$; we also write p_x for $p(x)$ and t_x for $t(x)$.

3.2 Contextual Categories

3.2.1 Original Definition

As defined in [1] and [2], a contextual category is defined to be a tree-structured category \mathbf{C} with the following additional structure:



in \mathbf{C} , an object f^*z such that $x \triangleleft f^*z$, a morphism $q(f, z) : f^*z \rightarrow z$ such that

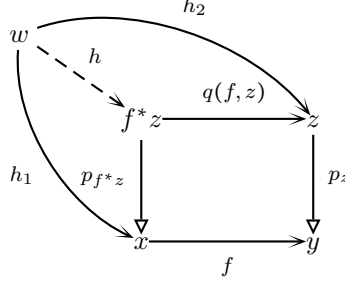
$$q(f, z) \circ p_z = p_{f^*z} \circ f \tag{q1}$$

i.e. such that the diagram:

$$\begin{array}{ccc}
 f^*z & \xrightarrow{q(f, z)} & z \\
 p_{f^*z} \downarrow & & \downarrow p_z \\
 x & \xrightarrow{f} & y
 \end{array}$$

commutes,

and, (ii), so that each such diagram is a pullback diagram, that is: for all objects w of \mathbf{C} , and for all morphisms $h_1 : w \rightarrow x$ and $h_2 : w \rightarrow z$ (see diagram 3.2.1 below) such that $h_1 \circ f = h_2 \circ p_z$ there exists a unique $h : w \rightarrow f^*z$ in \mathbf{C} such that $h \circ p_{f^*z} = h_1$ and $h \circ q(f, z) = h_2$, as shown here:



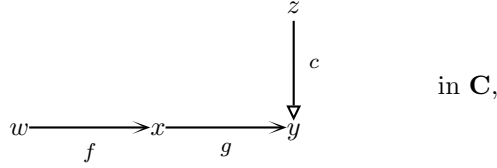
and so that (iii) whenever $x \triangleleft y$ in \mathbf{C} ,

$$id_x^* y = y \quad (\text{q2})$$

and

$$q(id_x, y) = id_y \quad (\text{q3})$$

and (iv) whenever



then

$$(f \circ g)^* z = f^*(g^* z) \quad (\text{q4})$$

and

$$q(f \circ g, z) = q(f, g^* z) \circ q(g, z) \quad (\text{q5})$$

3.2.2 Equational Definition following Voevodsky

Following Voevodsky, however, we may replace the pullback condition, (ii), above, by additional structure as follows:

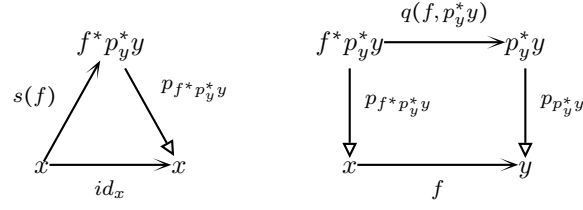
(ii') for all morphisms $f : x \rightarrow y$, a morphism $s(f) : x \rightarrow f^* p_y^* y$ such that:

$$s(f) \circ p_{f^* p_y^* y} = id_x \quad (\text{s1})$$

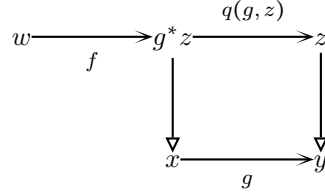
and

$$s(f) \circ q(f \circ p_y, y) = f \quad (\text{s2})$$

i.e. such that the following diagrams commute:



and such that whenever



in \mathbf{C} then

$$s(f \circ q(g, z)) = s(f) \quad (\text{s3})$$

This equation is well-typed because

$$\begin{aligned} lhs &= s(f \circ q(g, z)) \in \text{Hom}(w, (f \circ q(g, z))^* p_z^* z) \\ rhs &= s(f) \in \text{Hom}(w, (f \circ p_{g^*z})^* g^* z) \end{aligned}$$

and

$$\text{Hom}(w, (f \circ q(g, z))^* p_z^* z) = \text{Hom}(w, (f \circ p_{g^*z})^* g^* z)$$

because

$$\begin{aligned} (f \circ q(g, z))^* p_z^* z &= f^* ((q(g, z) \circ p_z)^* z) && \text{by (q4)} \\ &= f^* (p_{g^*z} \circ g)^* z && \text{by (q1)} \\ &= (f \circ p_{g^*z})^* g^* z && \text{by (q4)} \end{aligned}$$

3.2.3 Construction of pullbacks (Voevodsky)

Lemma 3.1. *If \mathbf{C} is a tree-structured category with operators, q , $*$ and s satisfying axioms q1-q5 and s1-s3 then for all objects w of \mathbf{C} , and for all morphisms $h_1 : w \rightarrow x$ and $h_2 : w \rightarrow z$ such that $h_1 \circ f = h_2 \circ p_z$ there exists a unique $h : w \rightarrow f^*z$ in \mathbf{C} such that $h \circ p_{f^*z} = h_1$ and $h \circ q(f, z) = h_2$.*

Proof. Use the following construction of such an h as given by Voevodsky (see diagram (2)):

$$h = s(h_2) \circ q(h_1, f^*z) \quad (1)$$

as shown in the following diagram:

$$\begin{array}{c}
(g_2 \circ p_z)^* z = g_1^* f^* z \\
\begin{array}{ccc}
s(g_2) \uparrow & & \\
\downarrow p_{g_1^* f^* z} & \searrow q(g_1, f^* z) & \\
w & \xrightarrow{h_2} & z \\
\downarrow h & \searrow q(f, z) & \\
f^* z & \xrightarrow{p_z} & y \\
\downarrow p_{f^* z} & & \\
x & \xrightarrow{f} & y
\end{array}
\end{array}
\quad (2)$$

As required we have

$$\begin{aligned}
h \circ p_{f^* z} &= s(h_2) \circ q(h_1, f^* z) \circ p_{f^* z} \\
&= s(h_2) \circ p_{h_1^* f^* z} \circ h_1 && \text{by (q1)} \\
&= s(h_2) \circ p_{(h_1 \circ f)^* z} \circ h_1 && \text{by (q4)} \\
&= s(h_2) \circ p_{(h_2 \circ p_z)^* z} \circ h_1 && \text{since } h_1 \circ f = h_2 \circ p_z \\
&= h_1 && \text{by (s1)}
\end{aligned}$$

and we have:

$$\begin{aligned}
h \circ q(f, z) &= s(h_2) \circ q(h_1, f^* z) \circ q(f, z) \\
&= s(h_2) \circ q(h_1 \circ f, z) && \text{by (q5)} \\
&= s(h_2) \circ q(h_2 \circ p_z, z) && \text{since } h_1 \circ f = h_2 \circ p_z \\
&= h_2 \circ id_w && \text{by (s2)} \\
&= h_2
\end{aligned}$$

To show that such an h is unique assume that we also have h' satisfying

$$h' \circ p_{f^* z} = h_1 \quad (3)$$

and

$$h' \circ q(f, z) = h_2 \quad (4)$$

then

$$\begin{aligned}
h' &= s(h') \circ q(h' \circ p_{f^* z}, f^* z) && \text{by (s2)} \\
&= s(h' \circ q(f, z)) \circ q(h' \circ p_{f^* z}, f^* z) && \text{by s3} \\
&= s(h \circ q(f, z)) \circ q(h \circ p_{f^* z}, f^* z) && \text{by (3) and (4)} \\
&= s(h) \circ q(h \circ p_{f^* z}, f^* z) && \text{by (s3)} \\
&= h && \text{by (s2)}
\end{aligned}$$

□

3.3 GAT of Contextual Categories

To summarise, the GAT of Contextual Categories is defined to be the GAT of Tree-Structured Categories plus the following additional structure:

Symbol Introductory Rule

$$\begin{array}{ll}
* & x \in Ob_n, y \in Ob_m, f \in Hom_{n,m}(x, y), z \in Ob_m(y) \vdash f^* z \in Ob_n(x) \\
q & x \in Ob_n, y \in Ob_m, f \in Hom_{n,m}(x, y), z \in Ob_m(y) \vdash q(f, z) \in Hom_{n,m+1}(f^* y_m, y_m) \\
s & x \in Ob_n, y \in Ob_m, f \in Hom_{n,m}(x, y) \vdash s(f) \in Hom_{n,n+1}(x, f^* p(y)^* y)
\end{array}$$

Axioms

$$\begin{array}{l}
q(f, z) \circ p_z = p_{f^* z} \circ f, \text{ whenever } x \in Ob_n, y \in Ob_m, f \in Hom_{n,m}(x, y), z \in Ob_m(y) \\
(f \circ g)^* z = f^*(g^* z), \text{ whenever } w \in Ob_n, x \in Ob_m, y \in Ob_p, f \in Hom_{n,m}(w, x), g \in Hom_{m,p}(x, y) \\
q(f \circ g, z) = q(f, g^* z) \circ q(g, z), \text{ whenever } w \in Ob_n, x \in Ob_m, y \in Ob_p, f \in Hom_{n,m}(w, x), g \in Hom_{m,p}(x, y) \\
s(f) \circ p_{f^* p_y^* y} = id(x), \text{ whenever } x \in Ob_n, y \in Ob_m, f \in Hom_{n,m}(x, y) \\
s(f) \circ q(f \circ p_y, y) = f, \text{ whenever } x \in Ob_n, y \in Ob_m, f \in Hom_{n,m}(x, y) \\
s(f \circ q(g, z)) = s(f), \text{ whenever } w \in Ob_n, x \in Ob_m, y \in Ob_p, g \in Hom_{m,p}(x, y), z \in Ob_{p+1}(y), f \in Hom_{n,m+1}(w, g^* z)
\end{array}$$

Before we go further we have one useful lemma:

Lemma 3.2 (lemma s4). *If \mathbf{C} is a contextual category then for all morphisms $f : x \rightarrow y$ and $g : y \rightarrow z$ in \mathbf{C} ,*

$$s(f \circ s(g)) = s(f \circ g)$$

Proof.

$$\begin{aligned}
s(f \circ s(g)) &= s(f \circ s(g) \circ q(g \circ p_z), z)) && \text{by axiom s3} \\
&= s(f \circ g) && \text{by axiom s2}
\end{aligned}$$

□

3.3.1 Extending the $*$ operator

As a preliminary to presenting an alternative axiomatisation of the generalised algebraic theory of contextual categories we introduce two new operators. The first we consider to be an extension to the $*$ operator and applies to certain morphisms as follows: if $f : x \rightarrow y_p$, $y_p \triangleleft y$, and $g : y_p \rightarrow y$ in a contextual category \mathbf{C} then define $f^* g : x \rightarrow (f \circ g \circ p_y)^* y$ to be the morphism $s(f \circ g)$.

Lemma 3.3. *If $f : x \rightarrow y_p$, $y_p \triangleleft y$, and $g : y_p \rightarrow y$ in a contextual category \mathbf{C} then $f^* g$ is a section i.e. $f^* g \circ p_{(f \circ g \circ p_y)^* y} = id_x$*

Proof. This follows from the definition of $*$ and from axiom s1. □

Lemma 3.4. *If $f : x \rightarrow y_p$ and $y_p \triangleleft y$ and $g : y_p \rightarrow y$ in (\mathbf{C}) as shown here:*

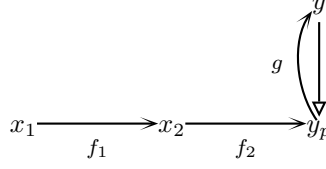
$$\begin{array}{ccc}
& & q(f \circ g \circ p_y, y) \\
& \nearrow & \longrightarrow \\
(f \circ g \circ p_y)^* y & \xrightarrow{\quad} & y \\
\downarrow f^* g & & \downarrow g \\
x & \xrightarrow{\quad f \quad} & y_p
\end{array}$$

then

$$f^* g \circ q(f \circ g \circ p_y, y) = f \circ g$$

Proof. Follows directly from the definition of $*$ and by axiom s2. □

Lemma 3.5. if $f_1 : x_1 \longrightarrow x_2$ and $f_2 : x_2 \longrightarrow y_p$ and $y_p \triangleleft y$ and $g : y_p \longrightarrow y$ in a contextual category \mathbf{C} as here:



then

$$f_1^* f_2^* g = (f_1 \circ f_2)^* g$$

Proof.

$$\begin{array}{ll}
 lhs = s(f_1 \circ f_2 \circ g) & \text{from defn. of extended } * \\
 = s(f_1 \circ s(f_2 \circ g)) & \text{by lemma 3.2} \\
 = f_1^* f_2^* g & \text{from defn. of extended } * \\
 = rhs & \text{as required.}
 \end{array}$$

□

3.3.2 Definition of δ

If x is an object of a contextual category \mathbf{C} then define $\delta_x : x \longrightarrow p_x^* x$ to be the morphism $s(id_x)$.

Lemma 3.6. If x is an object of a contextual category \mathbf{C} then δ_x satisfies:

- (i) δ_x is a section i.e $\delta_x \circ p_x = id_x$.
- (ii) $\delta_x \circ q(p_x, x) = id_x$.

Proof. Follows from definition of δ and from axioms s1 and axiom s3. □

Lemma 3.7. Whenever $y_p \triangleleft y$ and $f : x \longrightarrow y_p$ in a contextual category \mathbf{C} then

$$q(f, y)^* \delta_y = \delta_{f^* y}$$

Proof. Use lemma 3.2 and axiom s3. □

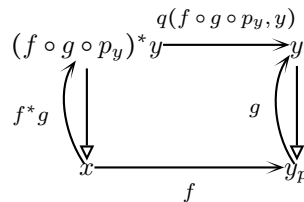
3.4 δ^* axiomatisation.

We can now give an alternative axiomatisation of the generalised algebraic theory of contextual categories which we provisionally call a δ^* -structure.

3.4.1 Definition

A δ^* -structure is a tree category (\mathbf{C}) with operators $*$ and q satisfying axioms q1 - q5 given above and with the following additional structure:

- I. If $f : x \longrightarrow y_p$ and $y_p \triangleleft y$ and $g : y_p \longrightarrow y$ in (\mathbf{C}) then a morphism $f^* g : x \longrightarrow (f \circ g \circ p_y)^* y$ as shown here:



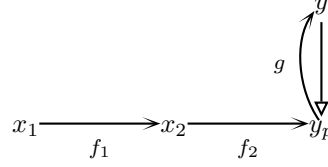
satisfying

$$f^* g \circ p_{(f \circ g \circ p_y)^* y} = id_x \quad (\text{star1})$$

and

$$f^* g \circ q(f \circ g \circ p_y, y) = f \circ g \quad (\text{star2})$$

and such that if $f_1 : x_1 \longrightarrow x_2$ and $f_1 : x_2 \longrightarrow y_p$ and $y_p \triangleleft y$ and $g : y_p \longrightarrow y$ in (C) as here:



then

$$f_1^* f_2^* g = (f_1 \circ f_2)^* g \quad (\text{star3})$$

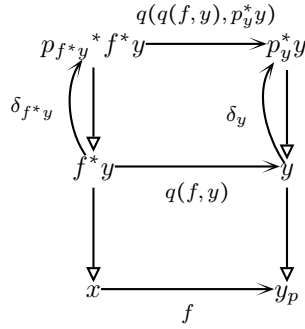
II. For any object x of (C) a morphism $\delta_x : x \longrightarrow p_x^* x$ such that δ_x is a section i.e :

$$\delta_x \circ p_x = id_x \quad (\text{delta1})$$

and

$$\delta_x \circ q(p_x, x) = id_x \quad (\text{delta2})$$

and whenever $y_p \triangleleft y$, $f : x \longrightarrow y_p$ in \mathbf{C} as here:



then

$$q(f, y)^* \delta_y = \text{delta}_{f^* y} \quad (\text{delta3})$$

3.4.2 Recovery of s from $*$ and delta

That a contextual category has δ^* -structure we have shown in lemmas 3.4, 3.5, 3.6 and 3.7. It remains to show that within any δ^* -structure an operator s can be defined and satisfies axioms s1, s2 and s3. Define $s(f)$ to be $f^* \delta_y$. That axiom s1 holds follows from axiom star1. We prove axiom s2, which is:

$$s(f) \circ q(f \circ p_y, y) = f,$$

as follows:

$$\begin{aligned}
lhs &= f^* \delta_y \circ q(f \circ p_y, y) && \text{from defn of s} \\
&= f^* \delta_y \circ q(f, p_y^* y) \circ q(p_y, y) && \text{by q5} \\
&= f \circ \delta_y \circ q(p_y, y) && \text{from star2} \\
&= f \circ id_y && \text{by delta2} \\
&= rhs && \text{as required.}
\end{aligned}$$

Finally we prove axiom s3, which is

$$s(f \circ q(g, z)) = s(f),$$

as follows:

$$\begin{aligned}
lhs &= (f \circ q(g, z))^* \delta_z && \text{from defn of s} \\
&= f^* (q(g, z)^* \delta_z) && \text{by star3} \\
&= f^* \delta_{g^* z} && \text{from delta3} \\
&= rhs && \text{as required.}
\end{aligned}$$

References

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